



INDIANA COMMON CORE STATE STANDARDS
ALGEBRA II

Introduction

The world we live in continues to change. For students to succeed in school, at work, and in the community, they will need more skills and knowledge than ever before. To ensure all students have every opportunity to succeed, Indiana adopted the Common Core State Standards in the area of Mathematics, as well as English Language Arts and Literacy in History/Social Studies, Science, and Technical Subjects.

Common Core State Standards Philosophy

The Common Core State Standards:

- are aligned with college and work expectations.
- include rigorous content *and* application of knowledge through high-order skills.
- build upon strengths and lessons of current state standards.
- are internationally benchmarked.
- are evidence and/or research-based.

The Common Core State Standards will

- help prepare all students with the knowledge and skills needed to succeed in college and careers.
- assist students who move between states.
- create the opportunity for America to compete for high-wage, high-skill jobs in a knowledge-based economy.
- allow for more focused pre-service education and professional development.
- create economies of scale around areas such as curriculum, instructional resources, and assessment.

Role of the Teacher

While the standards set expectations for student learning for each grade level, they do not prescribe how the standards should be taught. Teachers should use their skills, experience, talents, and resources to design standards-based classroom lessons that meet the needs of individual students.

Assessment Horizon

Indiana is a governing state in the Partnership for Assessment of Readiness for College and Careers (PARCC). Through PARCC a new generation of assessments will be developed and used throughout the nation, allowing us to compare and evaluate policies that affect student achievement across states and school districts.

The Common Core assessment includes a summative test, which will be administered online beginning with the 2014-15 school year. The assessment also includes “through-course assessments” that take place over the course of the year, allowing teachers to adjust their instruction based on students’ results. As a governing state, Indiana will pilot Common Core items in 2012-13 and 2013-14 and provide schools with data regarding student performance on the Common Core State Standards.

Impact on Instruction

Grades K, 1, and 2

In 2011-12, kindergarten teachers will be the first to teach *only* the Common Core. Those kindergarteners will be the first class of 3rd graders to participate in the Common Core assessment. First-grade and second-grade students will learn Indiana Academic Standards and identified elements of Common Core in 2011-12. First-grade students will learn *only* Common Core in 2012-13. Second-grade students will learn *only* Common Core in 2013-14.

Grades 3-12

Students in grades 3-12 will receive an intentional introduction to the Common Core beginning with the 2011-12 school year to ensure they have the foundation for future learning with no instructional gaps. In 2014-15, all ELA and math teachers will *only* teach Common Core.

Final Notes

Indiana Department of Education staff members are working diligently, both externally with other states, and internally across the IDOE, to ensure the development of a common scope and sequence for Common Core. This work is critical as we begin the transition to the Common Core State Standards.

Equally as important, efforts are underway to assist teachers in understanding what to teach next year (and in subsequent years). A tight alignment between the content which is “taught” and that which is “tested” on ISTEP+ and ECAs, together with the initiation of Common Core State Standards instruction, facilitates the important work of developing the foundation for future learning and assessment.

Traditional Pathway: Algebra II

Building on their work with linear, quadratic, and exponential functions, students extend their repertoire of functions to include polynomial, rational, and radical functions.² Students work closely with the expressions that define the functions, and continue to expand and hone their abilities to model situations and to solve equations, including solving quadratic equations over the set of complex numbers and solving exponential equations using the properties of logarithms. The Mathematical Practice Standards apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a coherent, useful, and logical subject that makes use of their ability to make sense of problem situations. The critical areas for this course, organized into four units, are as follows:

Critical Area 1: This unit develops the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Students connect multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers. Students identify zeros of polynomials, including complex zeros of quadratic polynomials, and make connections between zeros of polynomials and solutions of polynomial equations. The unit culminates with the fundamental theorem of algebra. A central theme of this unit is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

Critical Area 2: Building on their previous work with functions, and on their work with trigonometric ratios and circles in Geometry, students now use the coordinate plane to extend trigonometry to model periodic phenomena.

Critical Area 3: In this unit students synthesize and generalize what they have learned about a variety of function families. They extend their work with exponential functions to include solving exponential equations with logarithms. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect regardless of the type of the underlying function. They identify appropriate types of functions to model a situation, they adjust parameters to improve the model, and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. The description of modeling as *“the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions”* is at the heart of this unit. The narrative discussion and diagram of the modeling cycle should be considered when knowledge of functions, statistics, and geometry is applied in a modeling context.

Critical Area 4: In this unit, students see how the visual displays and summary statistics they learned in earlier grades relate to different types of data and to probability distributions. They identify different ways of collecting data—including sample surveys, experiments, and simulations—and the role that randomness and careful design play in the conclusions that can be drawn.

²In this course rational functions are limited to those whose numerators are of degree at most 1 and denominators of degree at most 2; radical functions are limited to square roots or cube roots of at most quadratic polynomials.

Units	Includes Standard Clusters*	Mathematical Practice Standards
Unit 1 Polynomial, Rational, and Radical Relationships	<ul style="list-style-type: none"> Perform arithmetic operations with complex numbers. Use complex numbers in polynomial identities and equations. Interpret the structure of expressions. Write expressions in equivalent forms to solve problems. Perform arithmetic operations on polynomials. Understand the relationship between zeros and factors of polynomials. Use polynomial identities to solve problems. Rewrite rational expressions. Understand solving equations as a process of reasoning and explain the reasoning. Represent and solve equations and inequalities graphically. Analyze functions using different representations. 	Make sense of problems and persevere in solving them. Reason abstractly and quantitatively. Construct viable arguments and critique the reasoning of others. Model with mathematics. Use appropriate tools strategically. Attend to precision. Look for and make use of structure. Look for and express regularity in repeated reasoning.
Unit 2 Trigonometric Functions	<ul style="list-style-type: none"> Extend the domain of trigonometric functions using the unit circle. Model periodic phenomena with trigonometric function. Prove and apply trigonometric identities. 	
Unit 3 Modeling with Functions	<ul style="list-style-type: none"> Create equations that describe numbers or relationships. Interpret functions that arise in applications in terms of a context. Analyze functions using different representations. Build a function that models a relationship between two quantities. Build new functions from existing functions. Construct and compare linear, quadratic, and exponential models and solve problems. 	
Unit 4 Inferences and Conclusions from Data	<ul style="list-style-type: none"> Summarize, represent, and interpret data on single count or measurement variable. Understand and evaluate random processes underlying statistical experiments. Make inferences and justify conclusions from sample surveys, experiments and observational studies. Use probability to evaluate outcomes of decisions. 	

*In some cases clusters appear in more than one unit within a course or in more than one course. Instructional notes will indicate how these standards grow over time. In some cases only certain standards within a cluster are included in a unit.

Unit 1: Polynomial, Rational, and Radical Relationships

This unit develops the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Students connect multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers. Students identify zeros of polynomials, including complex zeros of quadratic polynomials, and make connections between zeros of polynomials and solutions of polynomial equations. The unit culminates with the fundamental theorem of algebra. Rational numbers extend the arithmetic of integers by allowing division by all numbers except 0. Similarly, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme of this unit is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

Unit 1: Polynomial, Rational, and Radical Relationships	
Clusters and Instructional Notes	Common Core State Standards
<ul style="list-style-type: none"> Perform arithmetic operations with complex numbers. 	<p>N.CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.</p> <p>N.CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.</p>
<ul style="list-style-type: none"> Use complex numbers in polynomial identities and equations. <p><i>Limit to polynomials with real coefficients.</i></p>	<p>N.CN.7 Solve quadratic equations with real coefficients that have complex solutions.</p> <p>N.CN.8 (+) Extend polynomial identities to the complex numbers. <i>For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$.</i></p> <p>N.CN.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.</p>
<ul style="list-style-type: none"> Interpret the structure of expressions. <p><i>Extend to polynomial and rational expressions.</i></p>	<p>A.SSE.1 Interpret expressions that represent a quantity in terms of its context.*</p> <ol style="list-style-type: none"> Interpret parts of an expression, such as terms, factors, and coefficients. Interpret complicated expressions by viewing one or more of their parts as a single entity. <i>For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.</i> <p>A.SSE.2 Use the structure of an expression to identify ways to rewrite it. <i>For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.</i></p>
<ul style="list-style-type: none"> Write expressions in equivalent forms to solve problems. <p><i>Consider extending A.SSE.4 to infinite geometric series in curricular implementations of this course description.</i></p>	<p>A.SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. <i>For example, calculate mortgage payments.*</i></p>
<ul style="list-style-type: none"> Perform arithmetic operations on polynomials. <p><i>Extend beyond the quadratic polynomials found in Algebra I.</i></p>	<p>A.APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.</p>
<ul style="list-style-type: none"> Understand the relationship between zeros and factors of polynomials. 	<p>A.APR.2 Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.</p> <p>A.APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.</p>

Unit 1: Polynomial, Rational, and Radical Relationships	
Clusters and Instructional Notes	Common Core State Standards
<ul style="list-style-type: none"> Use polynomial identities to solve problems. <p><i>This cluster has many possibilities for optional enrichment, such as relating the example in A.APR.4 to the solution of the system $u^2+v^2=1$, $v = t(u+1)$, relating the Pascal triangle property of binomial coefficients to $(x+y)^{n+1} = (x+y)(x+y)^n$, deriving explicit formulas for the coefficients, or proving the binomial theorem by induction.</i></p>	<p>A.APR.4 Prove polynomial identities and use them to describe numerical relationships. For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.</p> <p>A.APR.5 (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.</p>
<ul style="list-style-type: none"> Rewrite rational expressions <p><i>The limitations on rational functions apply to the rational expressions in A.APR.6. A.APR.7 requires the general division algorithm for polynomials.</i></p>	<p>A.APR.6 Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.</p> <p>A.APR.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.</p>
<ul style="list-style-type: none"> Understand solving equations as a process of reasoning and explain the reasoning. <p><i>Extend to simple rational and radical equations.</i></p>	<p>A.REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.</p>
<ul style="list-style-type: none"> Represent and solve equations and inequalities graphically. <p><i>Include combinations of linear, polynomial, rational, radical, absolute value, and exponential functions.</i></p>	<p>A.REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*</p>
<ul style="list-style-type: none"> Analyze functions using different representations. <p><i>Relate F.IF.7c to the relationship between zeros of quadratic functions and their factored forms</i></p>	<p>F.IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*</p> <p>c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.</p>

Unit 2: Trigonometric Functions

Building on their previous work with functions, and on their work with trigonometric ratios and circles in Geometry, students now use the coordinate plane to extend trigonometry to model periodic phenomena.

Unit 2: Trigonometric Functions	
Clusters and Instructional Notes	Common Core State Standards
<ul style="list-style-type: none"> Extend the domain of trigonometric functions using the unit circle. 	<p>F.TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.</p> <p>F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.</p>
<ul style="list-style-type: none"> Model periodic phenomena with trigonometric functions. 	<p>F.TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.*</p>
<ul style="list-style-type: none"> Prove and apply trigonometric identities. <p><i>An Algebra II course with an additional focus on trigonometry could include the (+) standard F.TF.9: Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems. This could be limited to acute angles in Algebra II.</i></p>	<p>F.TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$, given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$, and the quadrant of the angle.</p>

Unit 3: Modeling with Functions

In this unit students synthesize and generalize what they have learned about a variety of function families. They extend their work with exponential functions to include solving exponential equations with logarithms. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect regardless of the type of the underlying function. They identify appropriate types of functions to model a situation, they adjust parameters to improve the model, and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. *The description of modeling as “the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions” is at the heart of this unit. The narrative discussion and diagram of the modeling cycle should be considered when knowledge of functions, statistics, and geometry is applied in a modeling context.*

Unit 3: Modeling with Functions	
Clusters and Instructional Notes	Common Core State Standards
<ul style="list-style-type: none"> Create equations that describe numbers or relationships. <p><i>For A.CED.1, use all available types of functions to create such equations, including root functions, but constrain to simple cases. While functions used in A.CED.2, 3, and 4 will often be linear, exponential, or quadratic the types of problems should draw from more complex situations than those addressed in Algebra I. For example, finding the equation of a line through a given point perpendicular to another line allows one to find the distance from a point to a line. Note that the example given for A.CED.4 applies to earlier instances of this standard, not to the current course.</i></p>	<p>A.CED.1 Create equations and inequalities in one variable and use them to solve problems. <i>Include equations arising from linear and quadratic functions, and simple rational and exponential functions.</i></p> <p>A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.</p> <p>A.CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. <i>For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.</i></p> <p>A.CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. <i>For example, rearrange Ohm’s law $V = IR$ to highlight resistance R.</i></p>
<ul style="list-style-type: none"> Interpret functions that arise in applications in terms of a context. <p><i>Emphasize the selection of a model function based on behavior of data and context.</i></p>	<p>F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. <i>Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*</i></p> <p>F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. <i>For example, if the function $h(n)$ gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*</i></p> <p>F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.*</p>

Unit 3: Modeling with Functions	
Clusters and Instructional Notes	Common Core State Standards
<ul style="list-style-type: none"> Analyze functions using different representations. <p><i>Focus on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate.</i></p>	<p>F.IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*</p> <ul style="list-style-type: none"> b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. <p>F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.</p> <p>F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.</p>
<ul style="list-style-type: none"> Build a function that models a relationship between two quantities. <p><i>Develop models for more complex or sophisticated situations than in previous courses.</i></p>	<p>F.BF.1 Write a function that describes a relationship between two quantities.*</p> <ul style="list-style-type: none"> b. Combine standard function types using arithmetic operations. <i>For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.</i>
<ul style="list-style-type: none"> Build new functions from existing functions. <p><i>Use transformations of functions to find models as students consider increasingly more complex situations.</i></p> <p><i>For F.BF.3, note the effect of multiple transformations on a single graph and the common effect of each transformation across function types.</i></p> <p><i>Extend F.BF.4a to simple rational, simple radical, and simple exponential functions; connect F.BF.4a to F.LE.4.</i></p>	<p>F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.</p> <p>F.BF.4 Find inverse functions.</p> <ul style="list-style-type: none"> a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. <i>For example, $f(x) = 2x^3$ or $f(x) = (x+1)/(x-1)$ for $x \neq 1$.</i>
<ul style="list-style-type: none"> Construct and compare linear, quadratic, and exponential models and solve problems. <p><i>Consider extending this unit to include the relationship between properties of logarithms and properties of exponents, such as the connection between the properties of exponents and the basic logarithm property that $\log xy = \log x + \log y$.</i></p>	<p>F.LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.</p>

Unit 4: Inferences and Conclusions from Data

In this unit, students see how the visual displays and summary statistics they learned in earlier grades relate to different types of data and to probability distributions. They identify different ways of collecting data—including sample surveys, experiments, and simulations—and the role that randomness and careful design play in the conclusions that can be drawn.

Unit 4: Inferences and Conclusions from Data	
Clusters and Instructional Notes	Common Core State Standards
<ul style="list-style-type: none"> Summarize, represent, and interpret data on a single count or measurement variable. <p><i>While students may have heard of the normal distribution, it is unlikely that they will have prior experience using it to make specific estimates. Build on students' understanding of data distributions to help them see how the normal distribution uses area to make estimates of frequencies (which can be expressed as probabilities). Emphasize that only some data are well described by a normal distribution.</i></p>	<p>S.ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.</p>
<ul style="list-style-type: none"> Understand and evaluate random processes underlying statistical experiments. <p><i>For S.IC.2, include comparing theoretical and empirical results to evaluate the effectiveness of a treatment.</i></p>	<p>S.IC.1 Understand statistics as a process for making inferences about population parameters based on a random sample from that population.</p> <p>S.IC.2 Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. <i>For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?</i></p>
<ul style="list-style-type: none"> Make inferences and justify conclusions from sample surveys, experiments, and observational studies. <p><i>In earlier grades, students are introduced to different ways of collecting data and use graphical displays and summary statistics to make comparisons. These ideas are revisited with a focus on how the way in which data is collected determines the scope and nature of the conclusions that can be drawn from that data. The concept of statistical significance is developed informally through simulation as meaning a result that is unlikely to have occurred solely as a result of random selection in sampling or random assignment in an experiment.</i></p> <p><i>For S.IC.4 and 5, focus on the variability of results from experiments—that is, focus on statistics as a way of dealing with, not eliminating, inherent randomness.</i></p>	<p>S.IC.3 Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.</p> <p>S.IC.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.</p> <p>S.IC.5 Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.</p> <p>S.IC.6 Evaluate reports based on data.</p>
<ul style="list-style-type: none"> Use probability to evaluate outcomes of decisions. <p><i>Extend to more complex probability models. Include situations such as those involving quality control, or diagnostic tests that yield both false positive and false negative results.</i></p>	<p>S.MD.6 (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).</p> <p>S.MD.7 (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).</p>

Mathematics | Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council’s report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy).

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions,

communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y .

8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.